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PERTURBATIONS OF COMETARY ORBITS

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1. Introduction

The orbits of over 500 comets have been calculated from observational data obtained while the comets were visible from the earth. From these observations it is found that the orbits fall into two main groups:

- (1) Those with parabolic or nearly parabolic orbits.
- (2) Those that have definitely elliptic orbits.

The second group contains all comets with periods less than 100 years. It is found that of over 70 comets that go to make up this group only three have an inclination to the ecliptic of more than 30° and the average inclination is 13° . Also, apart from Halley's comet, all have the same direction of motion around the sun as Jupiter. These two facts lead to the conjecture that originally all these comets belonged to the first group and have, by one or several close encounters with Jupiter, been diverted into their present orbits. Comet Brooks (1889 V) was observed to undergo this capture by Jupiter.

Both Lyttleton's [1] and Oort's [2] theories on the origin of the comets assume that the short-period comets were forced into their present orbits by the perturbative effects of the planets (in particular Jupiter) and that originally these comets were in osculating parabolic long-period orbits.

Van Woerkom [3] investigated the perturbations of comets which came near to Jupiter and he found that the root mean square values of the change in the reciprocal of the semimajor axis of comets with perihelion distances of 1 and 4.5 a.u. was 78 and $128 \text{ a.u.}^{-1} \times 10^{-5}$ respectively. The problem attempted in this paper is an extension of Van Woerkom's work. By a suitable choice of parameters the effect of Jupiter is evaluated for all parabolic comets whose distance of closest approach to the sun lies in the range 0.04 to 4 a.u. The lower limit of 0.04 a.u. is taken because Lyttleton's accretion theory postulates a stream of sun-grazing comets.

Jupiter's effect can be measured in terms of the change in the reciprocal of the semimajor axis of the orbit and an estimate of the distribution of this quantity is made for each value of the perihelion that is used.

Formerly with Manchester University Computing Machine Laboratory.

2. Calculation of the perturbations due to Jupiter

All the calculations are based on the energy changes as the comet describes a parabolic orbit.

If E is the energy per unit mass of a comet in describing a parabolic orbit about the sun, then the change in the energy δE for a complete orbit will be calculated. A sufficient gain in energy to make E positive will mean that the comet would escape from the solar system, and a negative E means that it will return in an elliptic orbit. In calculating the change in energy we will assume that Jupiter is the only planet that has a significant effect.

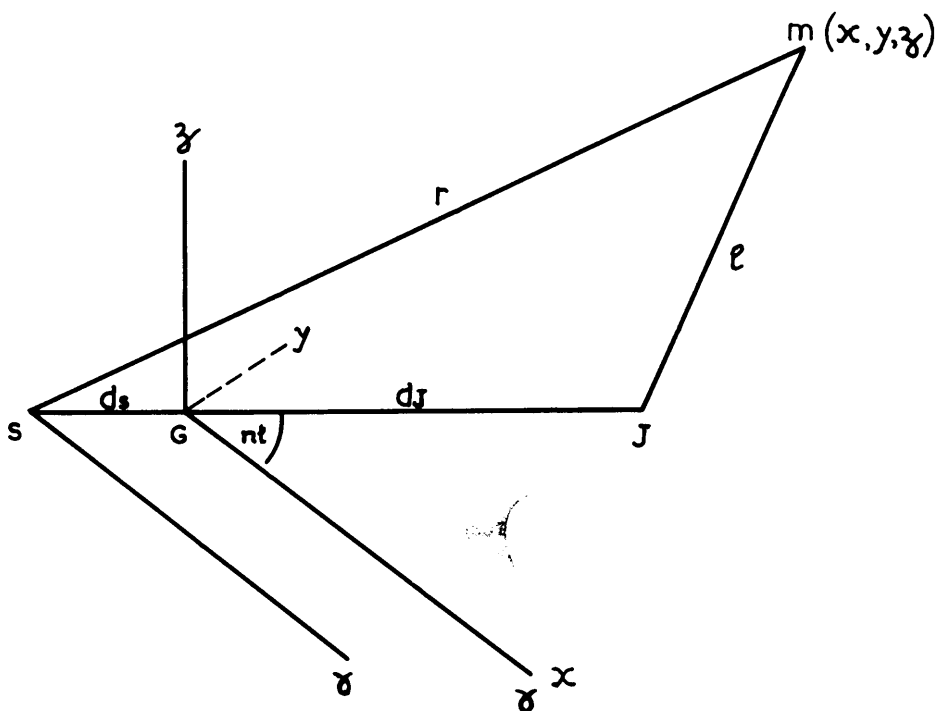


FIGURE 1

Comet in relation to sun and Jupiter.

The sun and Jupiter are assumed to be rotating about their common center of gravity G with constant angular velocity n . At any time t the comet is at a distance r from the sun and at a distance ρ from Jupiter (SG is the direction of Jupiter at $t = 0$). The energy E of the comet can then be obtained from the following considerations (using figure 1):

The comet m is at (x, y, z) (rotating axes as shown), so its equations of motion are

$$(2.1) \quad \ddot{x} - 2n\dot{y} - n^2x = G \frac{\partial}{\partial x} \left(\frac{S}{r} + \frac{J}{\rho} \right),$$

$$(2.2) \quad \ddot{y} + 2n\dot{x} - n^2y = G \frac{\partial}{\partial y} \left(\frac{S}{r} + \frac{J}{\rho} \right),$$

and

$$(2.3) \quad \ddot{z} = G \frac{\partial}{\partial z} \left(\frac{S}{r} + \frac{J}{\rho} \right),$$

where G is the gravitational constant and S and J are the masses of the sun and Jupiter respectively.

If we multiply (2.1) by \dot{x} , (2.2) by \dot{y} , (2.3) by \dot{z} , integrate, and then add together, we obtain Jacobi's energy integral

$$(2.4) \quad \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} n^2 (x^2 + y^2) = G \left(\frac{S}{r} + \frac{J}{\rho} \right) + C,$$

where C is a constant.

If, instead, m is referred to fixed axes at G , with X in the direction of Υ and Z coinciding with z , then the above integral becomes

$$(2.5) \quad \frac{1}{2} (\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - n(X\dot{Y} - \dot{X}Y) = G \left(\frac{S}{r} + \frac{J}{\rho} \right) + C,$$

since $\dot{X} = \dot{x} - n\dot{y}$ and $\dot{Y} = \dot{y} + n\dot{x}$. The last integral may be written,

$$(2.6) \quad E - nh_3 = \text{constant},$$

where $E = (1/2)v^2 - G(S/r + J/\rho) =$ total energy, and $h_3 = X\dot{Y} - \dot{X}Y =$ angular momentum about GZ .

Therefore, to find δE (the change in energy), it will be sufficient to find δh_3 .

The couple at G has its third component,

$$(2.7) \quad h_3 = \frac{GJ}{\rho^3} (XY_J - YX_J) + \frac{GS}{r^3} (XY_S - YX_S),$$

where the coordinates of the sun are $(X_S, Y_S, 0)$ and those of Jupiter are $(X_J, Y_J, 0)$.

Now let r, α, β be the spherical polar coordinates of m relative to parallel axes at S , that is, pole in the direction SZ , β measured from $S\Upsilon$; then

$$(2.8) \quad X + d_S \cos nt = r \sin \alpha \cos \beta,$$

$$(2.9) \quad Y + d_S \sin nt = r \sin \alpha \sin \beta,$$

$$(2.10) \quad Z = r \cos \alpha.$$

The coordinates of J and S are

$$(2.11) \quad \begin{aligned} X_J &= d_J \cos nt, & Y_J &= d_J \sin nt, & Z_J &= 0, \\ X_S &= -d_S \cos nt, & Y_S &= -d_S \sin nt, & Z_S &= 0. \end{aligned}$$

Therefore,

$$(2.12) \quad XY_J - YX_J = rd_J \sin \alpha \sin (nt - \beta)$$

and

$$(2.13) \quad XY_S - YX_S = -rd_S \sin \alpha \sin (nt - \beta);$$

and since $Sd_S = Jd_J = \mu$, say, the required component h_3 is

$$(2.14) \quad h_3 = \mu G \left(\frac{1}{\rho^3} - \frac{1}{r^3} \right) r \sin \alpha \sin (nt - \beta).$$

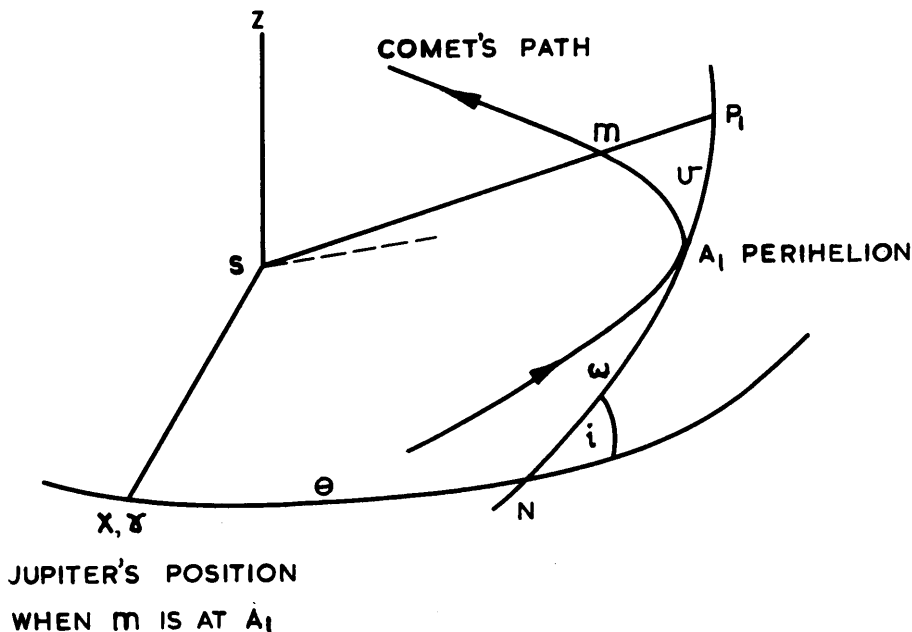


FIGURE 2

Comet's path about the sun.

Now suppose m is describing a parabola about S , which is not in the plane of SJ motion, then A_1 is the perihelion; A_1N is the great circle through A_1 which meets the ecliptic in N the ascending node, and

$$(2.15) \quad \begin{aligned} \Upsilon N &= \text{longitude of node} = \theta, \\ \pi - \Upsilon N A_1 &= \text{inclination} = i, \\ N A_1 &= \text{angular distance of perihelion from node} = \omega, \\ A_1 P_1 &= v. \end{aligned}$$

Note the direction $S\Upsilon$ is the direction of Jupiter at time $t = 0$. The direction cosines of SP_1 give

$$\begin{aligned}
 & \cos \theta \cos (\omega + v) - \sin \theta \sin (\omega + v) \cos i = \sin \alpha \cos \beta, \\
 (2.16) \quad & \sin \theta \cos (\omega + v) + \cos \theta \sin (\omega + v) \cos i = \sin \alpha \sin \beta, \\
 & \sin (\omega + v) \sin i = \cos \alpha.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (2.17) \quad & \sin \alpha \cos (nt - \beta) \\
 & = \cos (\omega + v) \sin (nt - \theta) - \sin (\omega + v) \cos i \cos (nt - \theta);
 \end{aligned}$$

therefore

$$\begin{aligned}
 (2.18) \quad & \delta E = n \delta h_3 \\
 & = n G \mu \int_{-\infty}^{\infty} \left(\frac{1}{\rho^3} - \frac{1}{r^3} \right) r f(t) dt,
 \end{aligned}$$

where

$$(2.19)$$

$$\begin{aligned}
 f(t) &= \cos (\omega + v) \sin (nt - \theta) - \sin (\omega + v) \cos (nt - \theta) \cos i, \\
 \rho^2 &= d^2 + r^2 - 2dr[\cos (\omega + v) \cos (nt - \theta) + \sin (\omega + v) \sin (nt - \theta) \cos i],
 \end{aligned}$$

and $d = d_s + d_J$ (that is, d is the distance between the sun and Jupiter).

The equation of the parabola that m is describing about S is

$$(2.20) \quad r = q \sec^2 \frac{v}{2},$$

where $q = SA_1$ (that is, distance of closest approach).

It should be noted that if the comet were describing an ellipse or a hyperbola, then the equation of the orbit would contain an extra parameter, that is, the eccentricity of the orbit.

If we let $\tau = \tan v/2$, then (2.20) becomes

$$\begin{aligned}
 (2.21) \quad & r = q(1 + \tau^2), \\
 & d\tau = \frac{1}{2} \sec^2 \frac{v}{2} dv.
 \end{aligned}$$

Now, for the undisturbed parabolic orbit about the sun we have by Kepler's laws

$$\begin{aligned}
 (2.22) \quad & t = \left(\frac{2q^3}{S} \right)^{1/2} \int \frac{dv}{(1 + \cos v)^2} \\
 & = \left(\frac{2q^3}{S} \right)^{1/2} \int (1 + \tau^2) d\tau \\
 & = \left(\frac{2q^3}{S} \right)^{1/2} \left(\tau + \frac{1}{3} \tau^3 \right)
 \end{aligned}$$

if we take $\tau = 0$ when $t = 0$.

Also, since Jupiter is in a circular orbit about the sun, we have $S/d^2 = n^2 d_J$.

Thus $(S + J)/d = n^2 d^2$, since $S/d_J = J/d_S = (S + J)/d$. Therefore, $n^2 = (S + J)/d^3$.

Let

$$(2.23) \quad \begin{aligned} \lambda &= n \left(\frac{2q^3}{S} \right)^{1/2} \\ &= \left(\frac{2q^3}{S} \frac{S + J}{d^3} \right)^{1/2}, \end{aligned}$$

but since J is negligible in comparison with S we may make the approximation

$$(2.24) \quad \lambda = \left(\frac{2q^3}{d^3} \right)^{1/2}.$$

It is normal to express the perturbative effects not in terms of the energy change but in terms of the change in the reciprocal of the semimajor axis of the comet's orbit, that is, $\delta(1/a)$. Since the binding energy of a nearly parabolic orbit is $GS/2a$, where G is the gravitational constant, S is the mass of the sun, and a is the semimajor axis of the comet's orbit, we have

$$(2.25) \quad \begin{aligned} \delta \left(\frac{1}{a} \right) &= -\frac{2}{GS} \delta E \\ &= \frac{2n\mu}{S} \int_{-\infty}^{\infty} \left(\frac{1}{r^3} - \frac{1}{\rho^3} \right) r f(t) dt, \end{aligned}$$

where

$$(2.26) \quad \mu = \frac{SJ}{(S + J)d} = Sd_S = Jd_J.$$

3. Numerical evaluation of $\delta(1/a)$

To evaluate $\delta(1/a)$ we break the integral into two parts,

$$(3.1) \quad I_S = \frac{2n\mu}{S} \int_{-\infty}^{\infty} \frac{f(t)}{r^2} dt$$

and

$$(3.2) \quad I_J = \frac{-2n\mu}{S} \int_{-\infty}^{\infty} \frac{rf(t)}{\rho^3} dt.$$

If we make the substitution $nt = \lambda[\tau + (1/3)\tau^3]$ in I_S and use the relationships $r = q(1 + \tau^2)$ and $\tau = \tan v/2$, we get

$$(3.3) \quad I_S = \frac{4\lambda\mu}{Sq^2} \int_0^{\infty} \frac{1}{(1 + \tau^2)^2} f_1(\tau) d\tau,$$

where

$$(3.4) \quad \begin{aligned} f_1(\tau) &= (1 - \tau^2) \cos nt(\cos \omega \sin \theta + \sin \omega \cos \theta \cos i) \\ &\quad + 2\tau \sin nt(\sin \omega \cos \theta + \cos \omega \sin \theta \cos i). \end{aligned}$$

Hence the parameters ω , θ , and i can be separated out and the problem is now reduced to evaluating the following two integrals,

$$(3.5) \quad I_1 = \text{Imag} \int_0^\infty \frac{2}{(1 + \tau^2)^2} \exp \left[i\lambda \left(\tau + \frac{1}{3} \tau^3 \right) \right] d\tau,$$

and

$$(3.6) \quad I_2 = \text{Real} \int_0^\infty \frac{1 - \tau^2}{(1 + \tau^2)^2} \exp \left[i\lambda \left(\tau + \frac{1}{3} \tau^3 \right) \right] d\tau.$$

If the path of integration is taken as the contour $\tau = r \exp (\pi/6i)$ the integrands die away very rapidly and the evaluation is trivial. These integrals were evaluated on the Manchester University Ferranti Mercury Computer for the following values of λ :

$$(3.7) \quad \lambda = 0.001 \ (0.001) \ 0.01 \ (0.01) \ 0.1 \ (0.1) \ 1$$

(when $\lambda = 0.001$, $q \approx 0.04$ a.u.).

The method of evaluation used was that of Gaussian quadrature, which is incorporated in a standard library program for infinite integrals [4].

The integration of I_J , however, is not as simple as this because of the presence of branch points of the integrand in the complex τ plane. If $\tau = u + iv$ then it can be shown that for large τ these branch points lie on the curves

$$(3.8) \quad v = \pm \frac{1}{\lambda(1 + u^2)} \log \left(\frac{i + u^2}{k} \right),$$

and their positions on these curves are given by

$$(3.9) \quad u = \frac{3(2n \pm 1)\pi + \phi}{\lambda},$$

where ϕ and k are constants which depend on the parameters of the comet's orbit.

Because of these branch points it is impossible to deform the contour as was done in the case of I_S .

If we make the substitutions

$$(3.10) \quad \begin{aligned} nt = x &= \lambda \left(\tau + \frac{1}{3} \tau^3 \right), \\ \tau &= \tan \frac{v}{2}, \\ r &= q(1 + \tau^2), \end{aligned}$$

then

$$(3.11) \quad I_J = \frac{-2\mu}{Sd^2} \left(\frac{\lambda^2}{2} \right)^{-1/6} \int_{-\infty}^{\infty} \frac{f_2(x)}{g_2(x)} dx,$$

where $f_2(x) = f_1(\tau)$

and

$$(3.12) \quad g_2(x) = (b^{-1} + b(1 + \tau^2)^2 - 2\{\cos(x - \theta)[(1 - \tau^2)\cos\omega - 2\tau\sin\omega] \\ + \cos i \sin(x - \theta)[(1 - \tau^2)\sin\omega + 2\tau\cos\omega]\})^{3/2},$$

where $b = \lambda/\sqrt{2}$.

It can be shown that for sufficiently large x_0

$$(3.13) \quad \int_{x_0}^{\infty} \frac{f_2(x)}{g_2(x)} dx \sim \left(\frac{\lambda}{3}\right)^{4/3} \int_{x_0}^{\infty} \frac{\sin\omega \cos i \cos x - \cos\omega \sin x}{x^{4/3}} dx \\ + \left(\frac{\lambda}{3}\right)^{5/3} \int_{x_0}^{\infty} \frac{\sin\omega \sin x + \cos\omega \cos x \cos i}{x^{5/3}} dx \\ + \int_{x_0}^{\infty} O\left(\frac{\cos x}{x^2}\right) dx;$$

but

$$(3.14) \quad \int_{x_0}^{\infty} \frac{\cos x}{x^r} dx = O(x_0^{-r}).$$

Therefore the remaining terms of the asymptotic expansion are $O(x_0^{-2})$.

It will also be noticed that the sufficiently large x_0 the terms in the asymptotic expansion behave like

$$(3.15) \quad \int_{x_0}^{\infty} f(x) \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} dx.$$

The method that suggests itself from this is that we integrate directly as far out as x_0 and then from x_0 onward we integrate in steps of π and the contributions from each step are added to the value of the integral so far obtained. In this way an alternating sequence is formed and the limit of the sequence can be estimated by the standard Euler method. It was found that for values of $x_0 > 20\pi$, the integrand showed sufficient oscillatory character for the Euler technique to be used.

If $\delta(1/a)$ is considered as a function of the parameters θ and ω then the following relationships hold,

$$(3.16) \quad \delta\left(\frac{1}{a}\right)[\theta, \omega] = \delta\left(\frac{1}{a}\right)[\theta + \pi, \omega + \pi], \\ \delta\left(\frac{1}{a}\right)[\theta, \omega] = -\delta\left(\frac{1}{a}\right)[\pi - \theta, \pi - \omega],$$

and when $i = 0$ and π , the parameters θ and ω can be combined as one parameter ϕ . Therefore if a step of $\pi/8$ is taken in each of the parameters i , θ , and ω , the following values are sufficient to cover the sphere: $i = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8$, $\theta = 0, \pi/8, 2\pi/8, 3\pi/8, 4\pi/8, 5\pi/8, 6\pi/8, 7\pi/8$, $\omega = 0, \pi/8, 2\pi/8, 3\pi/8, 4\pi/8, 5\pi/8, 6\pi/8, 7\pi/8$. The values of λ used are as in the evaluation of I_s , where $\lambda = (2q^3/d^3)^{1/2}$, and where q is the perihelion distance and d is the distance between the sun and Jupiter.

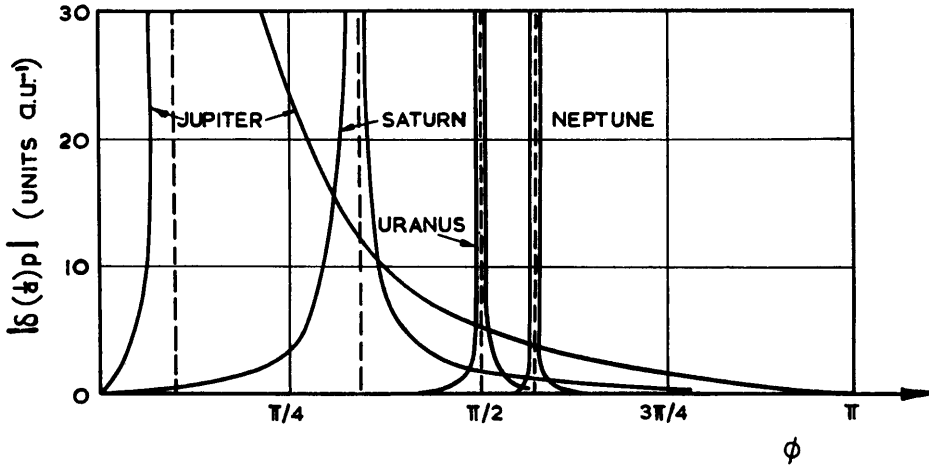


FIGURE 3

Graph of $|\delta(1/a)_p|$ as a function of ϕ
for the case perihelion distance, $q = 4.130$ a.u.,
that is, for Jupiter $\lambda = 1$.

4. Results

In calculating the energy change it was assumed that the effect of the other planets was small when compared with Jupiter's effect. To show this, similar calculations to the above were performed for Saturn, Uranus, and Neptune and the energy changes produced by these four major planets were compared. The

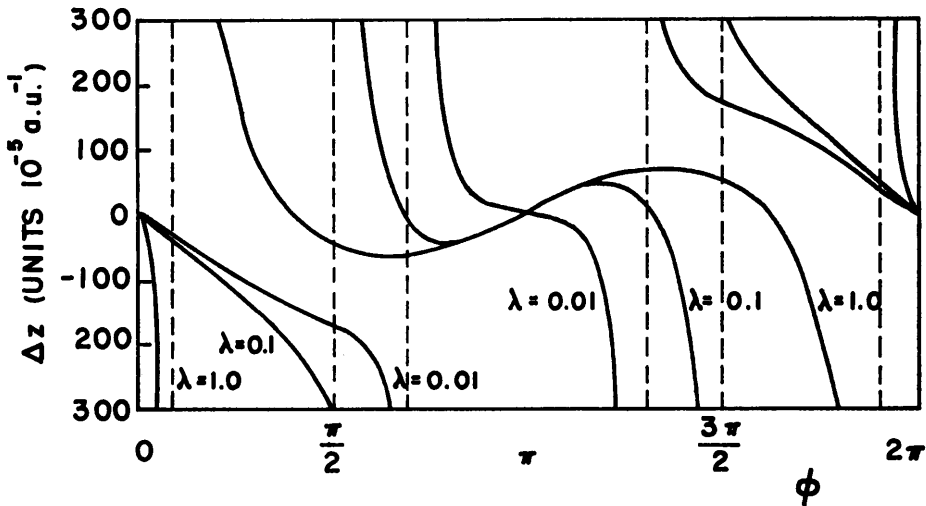


FIGURE 4

Graph of Δz as a function of ϕ
for the case $i = 0$ and $\lambda = 0.01, 0.1$ and 1.0 .

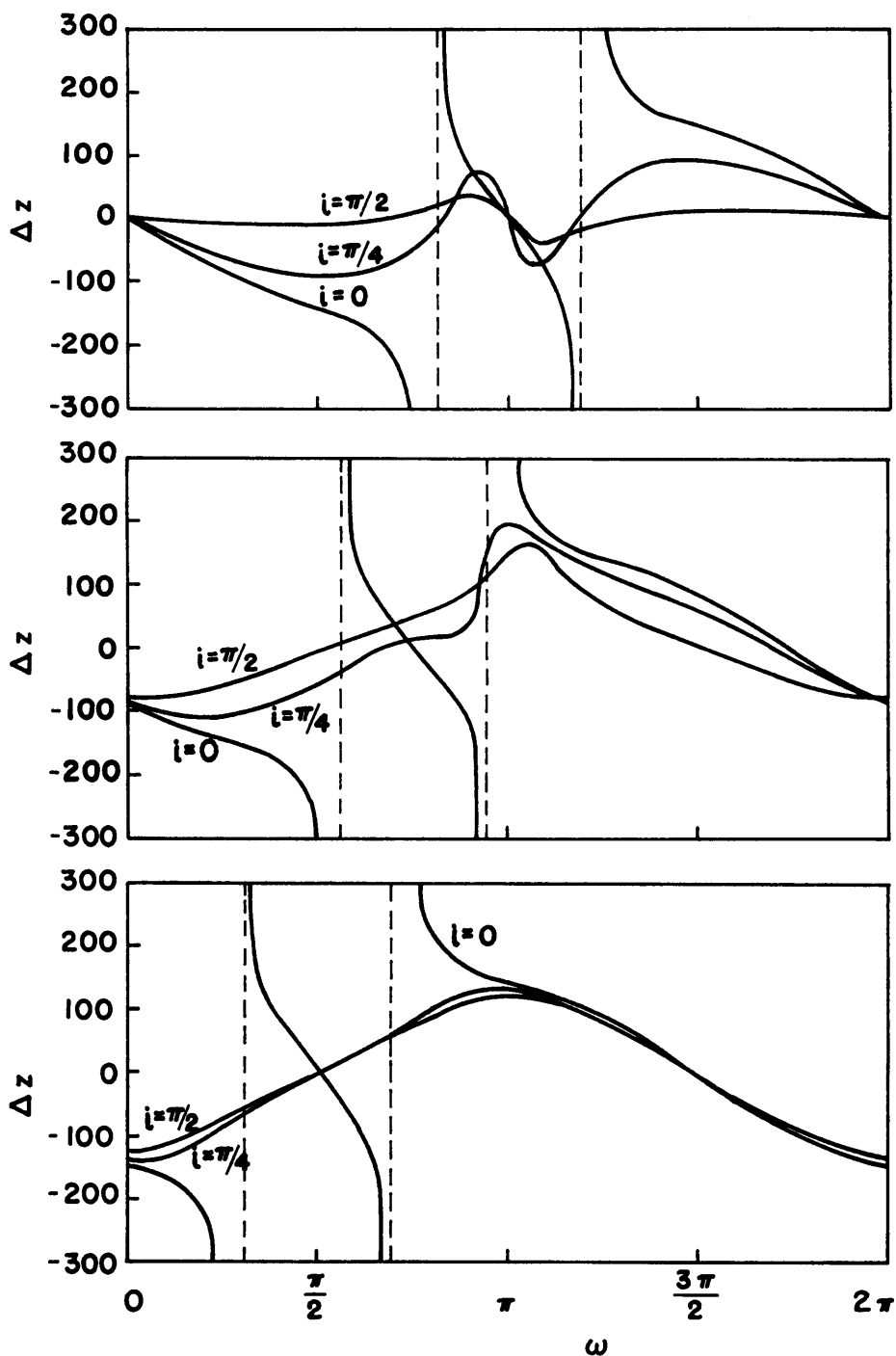


FIGURE 5

Graph of Δz as a function of ω
 for the case $\lambda = 0.001$ and $i = 0, \pi/4$ and $\pi/2$.
Upper panel: $\theta = 0$; *middle panel:* $\theta = \pi/4$; *lower panel:* $\theta = \pi/2$.

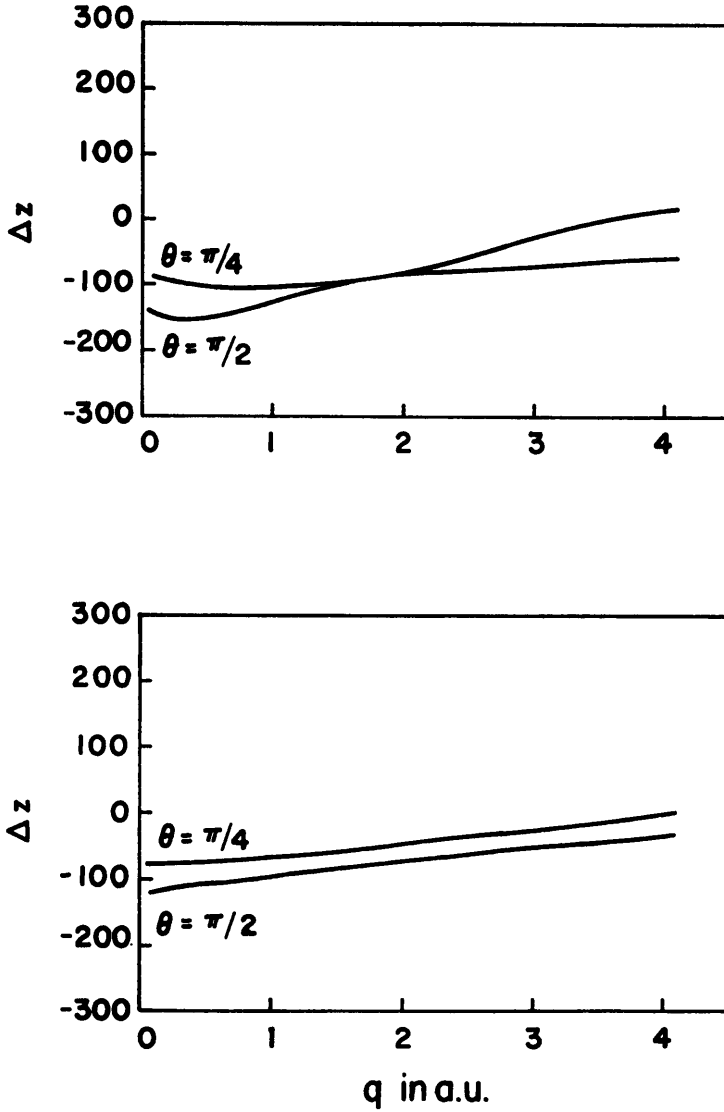


FIGURE 6

Graph of Δz as a function of q
 for the case $\omega = 0$, $\theta = \pi/4$ and $\pi/2$.
Upper panel: $i = \pi/4$; lower panel: $i = \pi/2$.

quantity $\delta(1/a)p$ was evaluated for them for major planets for $i = 0$, $\phi = 0(\pi/8)\pi$, and for a perihelion distance of 4.130 a.u. (this corresponds to $\lambda = 1$ in the case of Jupiter). The $i = 0$ plane was chosen because there is always a singularity of I_J in this plane and hence $\delta(1/a)p$ will always obtain its maximum, that is, $\delta(1/a)p$ will become infinite when the comet collides with the planet.

The results are set out in graphical form in figure 3 and for convenience $|\delta(1/a)p|$ was plotted against ϕ . It will be seen that the effect of Uranus and Neptune is negligible except when the comet is nearly in collision with these planets. However, Saturn's effect, while smaller than Jupiter's, is significant, and in ignoring it we have introduced errors which can be large in the case of comets which pass close to Saturn. In calculating these curves it was assumed that the planets moved in circular orbits.

Typical results are set out in figures 4, 5, and 6. For convenience the substitution $\Delta z = 10^5 \delta(1/a)$ has been made. From these it will be seen that not only does $|\Delta z|$ depend on how close the comet comes to Jupiter but also on the angle at which the planes cut, that is, as i tends to $\pi/2$ the magnitude Δz decreases. This remark, however, is not always true because there are singularities of I_J other than in the $i = 0$ plane (for example, there is one at approximately $i = \pi/2$, $\theta = 7\pi/8$, $\omega = 10\pi/8$, and $\lambda = 1$), and the effect of these will tend to invalidate any general comments.

5. Calculation on particular comets

The program was also used to evaluate Δz for six of the comets that have been observed to be hyperbolic at their perihelion. The effects of both Jupiter and Saturn were taken into account. In making these calculations it was assumed that the comets were in parabolic orbits, and that the planets were moving in circular orbits around the sun. Galibina [5] has also calculated Δz for 20 comets and where the two lists overlap her results have been included in table I (which gives the post orbit) for comparison. From these results it will be noted that only Comet 1914 V will return.

TABLE I

Comet	Δz units 10^{-5} a.u. $^{-1}$	Δz (Galibina)
1914 V	3.69	3.8
1922 II	-58.86	—
1925 I	-63.12	-59.1
1925 VII	-32.55	—
1932 VI	-27.56	-26.1
1936 I	-35.95	-49.1

Van Biesbroeck [6] has also calculated Δz for Comet 1914 V and he obtained a value of 12.6.

6. Distribution of Δz [$\Delta z = 10^5 \delta(1/a)$]

If it is supposed that the parabolic comets for a given λ (that is, a given perihelion distance q) have their orbits disposed at random then we require the

distribution function for Δz . Consider the coordinates (i, θ) which determine the plane of the comet's orbit. Then if dS is an element of area on the celestial sphere we have

$$(6.1) \quad dS = C \sin i \, d\theta \, di,$$

where C is a normalizing factor which depends on the range of variables.

Therefore, since ω is uniformly distributed, the distribution of Δz is given by

$$(6.2) \quad C \sin i \, di \, d\theta \, d\omega = -C \, d(\cos i) \, d\theta \, d\omega.$$

This suggests that we ought to make $\cos i$, θ , and ω independently uniformly distributed. Therefore in the above calculations it is necessary to weight the values of Δz associated with each i in such a way that the distribution of $\cos i$ is uniform.

Now Δz has been evaluated for nine values of i given by i_r where $i_r = r\pi/8$ with $r = 0(1)8$; then the weight to be associated with each of these values of i_r is

TABLE II
VALUES OF q AND σ CORRESPONDING
TO 28 VALUES OF λ

λ	q (in a.u.)	σ
0.001	0.0413	77.9
0.002	0.0656	77.2
0.003	0.0859	77.8
0.004	0.1041	78.7
0.005	0.1207	80.7
0.006	0.1364	81.2
0.007	0.1511	81.3
0.008	0.1652	79.2
0.009	0.1787	78.2
0.010	0.1917	77.7
0.020	0.3043	79.2
0.030	0.3987	75.8
0.040	0.4830	75.7
0.050	0.5605	76.5
0.060	0.6329	78.9
0.070	0.7014	80.6
0.080	0.7667	77.3
0.090	0.8293	75.0
0.100	0.8897	73.1
0.200	1.4123	68.9
0.300	1.8506	60.6
0.400	2.2418	55.5
0.500	2.6014	52.6
0.600	2.9376	47.7
0.700	3.2556	46.8
0.800	3.5587	50.1
0.900	3.8494	47.8
1.000	4.1295	49.3

$$(6.3) \quad \begin{aligned} & \frac{\cos i_{r+1} - \cos i_{r-1}}{2}, & r \neq 0 \text{ or } 8, \\ & \frac{\cos i_1 - \cos i_0}{2}, & r = 0 \text{ or } 8. \end{aligned}$$

Therefore each triple $\Delta z(i, \theta, \omega)$ is given a weight according to the value of i associated with it. The r.m.s. value Δz is therefore the square root of $\sum w(\Delta z)^2 / \sum w$, where $\sum w = 128$.

Now let this r.m.s. value be σ ; then table II gives the values of σ and q corresponding to the 28 values of λ . These values are also plotted against q in figure 7 and a straight line, in the sense of least squares, is fitted. The line obtained is

$$(6.4) \quad \sigma = -9.24q + 80.8$$

and the sum of the squares of the residuals = 263.

If this formula is applied to calculate σ for the two cases given by Van Woerkom [3], that is, $q = 4.5$ a.u. and $q = 1$ a.u., we obtain the following results,

$$(6.5) \quad \begin{aligned} \sigma &= 42 \quad \text{when } q = 4.5 \text{ a.u.}, \\ \sigma &= 72 \quad \text{when } q = 1 \text{ a.u.}, \end{aligned}$$

which compare with Van Woerkom's results of

$$(6.6) \quad \begin{aligned} \sigma &= 128 \quad \text{when } q = 4.5 \text{ a.u.}, \\ \sigma &= 78 \quad \text{when } q = 1 \text{ a.u.} \end{aligned}$$

The distribution of Δz is found by summing the weights of all the triples $\Delta z(i, \theta, \omega)$ which give values of Δz in each of the intervals $-300(10)300$ in units of $10^{-6}(\text{a.u.})^{-1}$. The contribution from each interval is then divided by 128 to give

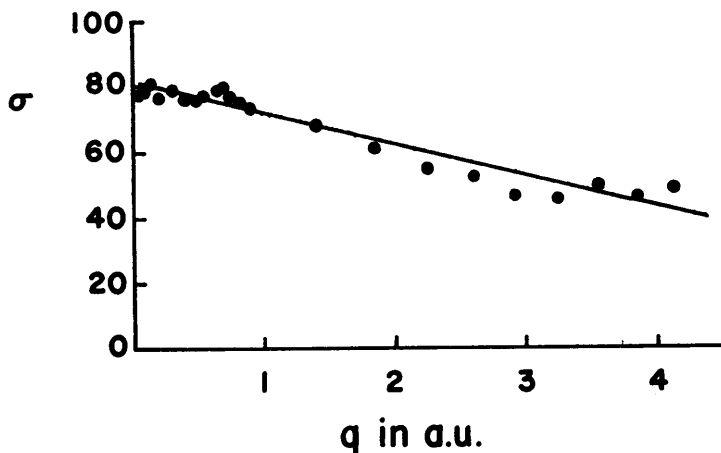


FIGURE 7

Plot of σ as a function of q .
The fitted line is $\sigma = -9.24q + 80.8$.

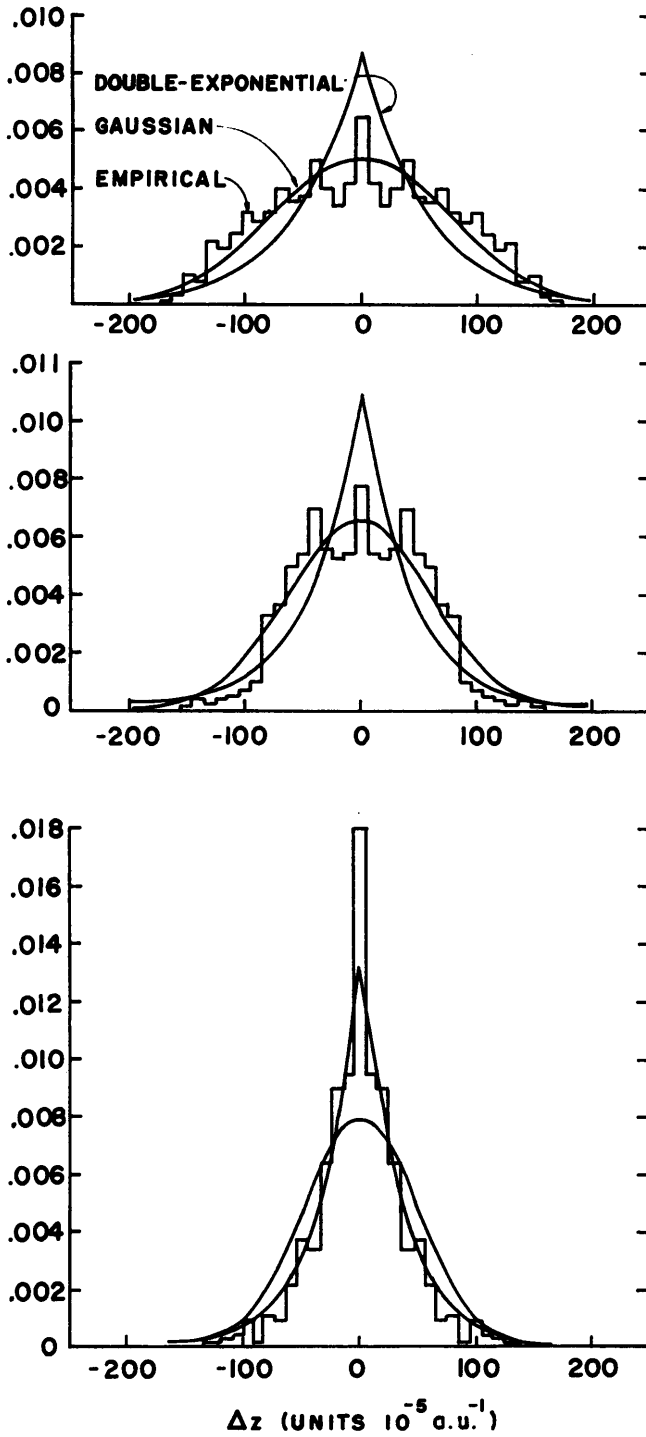


FIGURE 8

Comparison of double-exponential, Gaussian, and empirical distribution of Δz .

Upper panel: $q = 0.0413 \text{ a.u.}$, $\sigma = 77.9$;

middle panel: $q = 1.8506 \text{ a.u.}$, $\sigma = 60.6$;

lower panel: $q = 4.1295 \text{ a.u.}$, $\sigma = 49.3$

the fraction of the unit probability associated with that interval. Typical distributions for $q = 0.0413$ a.u. ($\lambda = 0.001$), $q = 1.8506$ a.u. ($\lambda = 0.300$), and $q = 4.1295$ a.u. ($\lambda = 1.000$) are given in figure 8 and these distributions are compared graphically with the normal (Gaussian) distribution, having the same mean and variance (that is, σ^2) and $(b/2) \exp(-|\Delta z|/b) d(\Delta z)$, where $\sigma = \sqrt{2}b$, which is the double exponential distribution used by Kendall [7] in some of his theoretical work.

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